A new class of micro–macro models for elastic–viscoplastic heterogeneous materials

Hafid Sabar [a], Marcel Berveiller [a,*], Veronique Favier [a,*], Stephane Berbenni [a,b]

[a] *Laboratoire de Physique et Mécanique des Matériaux, UMR CNRS 7554, Institut Supérieur de Génie Mécanique et Productique, Ile du Saulcy, 57045 Metz Cedex 1, France*
[b] *Usinor Research and Development, SOLLAC, 17 avenue des Tilleuls, 57191 Florange Cedex, France*

This paper is dedicated to Prof. Dr. rer. nat. Ekkehart Kröner's memory

Abstract

The determination of the effective behavior of heterogeneous materials from the properties of the components and the microstructure constitutes a major task in the design of new materials and the modeling of their mechanical behavior. In real heterogeneous materials, the simultaneous presence of instantaneous mechanisms (elasticity) and time dependent ones (non-linear viscoplasticity) leads to a complex space–time coupling between the mechanical fields, difficult to represent in a simple and efficient way. In this work, a new self-consistent model is proposed, starting from the integral equation for a translated strain rate field. The chosen translated field is the (compatible) viscoplastic strain rate of the (fictitious) viscoplastic heterogeneous medium submitted to a uniform (unknown) boundary condition. The self-consistency condition allows to define these boundary conditions so that a relative simple and compact strain rate concentration equation is obtained. This equation is explained in terms of interactions between an inclusion and a matrix, which lead to interesting conclusions. The model is first applied to the case of two-phase composites with isotropic, linear and incompressible viscoelastic properties. In that case, an exact self-consistent solution using the Laplace–Carson transform is available. The agreement between both approaches appears quite good. Results for elastic–viscoplastic BCC polycrystals are also presented and compared with results obtained from Kröner–Weng's and Paquin et al. (Arch. Appl. Mech. 69 (1999) 14)'s model. © 2002 Elsevier Science Ltd. All rights reserved.

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1. Introduction

The self-consistent method, originally proposed by Hershey (1954) and Kröner (1958) for heterogeneous materials with linear elastic behavior, was extended later to behavior of incremental elastic–plastic (Hill, 1965) and viscous (Hutchinson, 1976) types. Another model suggested by Kröner (1961) relates to the

* Corresponding authors. Tel.: +33-3-8737-5430; fax: +33-3-8737-5470.
 E-mail address: veronique.favier@metz.ensam.fr (V. Favier).

calculation of internal stresses due to an incompatible inelastic strain field (problem of the plastic inclusion embedded in a plastic matrix). The resulting interaction law was awkwardly used to build scale transition schemes that proved to be too stiff in the calculation of the interactions between the components of the heterogeneous material (Berveiller and Zaoui, 1979).

The cases of viscoelasticity and elastic-viscoplasticity raise a more complicated problem due mainly to the differential nature of the constitutive equations that involve different orders of time derivation concerning strain and/or stress fields.

Complex couplings between the mechanical fields result from the hereditary nature of the behavior and the heterogeneous character of the material so that long memory effect appear: the behavior of the representative volume element (RVE) formed by Maxwell elements cannot be represented by a Maxwell solid.

The modeling of such complex systems can be based on "hereditary approaches" using (for linear materials following Boltzman's superposition principle) a time integral formulation. For elastic–viscoplastic non-linear problems, a tangent linearization procedure (Rougier et al., 1993) along the loading path leads to a viscoelastic one with eigenstrains which can be solved using classical Laplace–Carson transform. Li and Weng (1997) propose a secant-viscosity approach making use of a linear viscoelastic comparison medium. In the case of isotropic and incompressible viscoplastic properties, the secant viscosity is reduced to a scalar. This secant viscosity at a given stage of deformation can be identified to a constant viscosity of a linear Maxwell comparison material. Then, to determine the effective properties of this viscoelastic comparison composite, Hashin's model (1969) using the Laplace transform is applied. As the secant viscosity of the ductile matrix changes continuously, the properties of the viscoelastic comparison composite is developed in rate forms so that the shear viscosity of the Maxwell matrix is continuously adjusted. The application of the model in the case of a creep test allows to easily return to the real space. But, in general, the methods using Laplace–Carson transform require large CPU time and memory space and are not well adapted for non-linear situations; moreover inversions are not easy to find except in creep behavior (Masson and Zaoui, 1999; Li and Weng, 1997).

Internal variable approaches can be preferred for their simplicity and their natural formulation and expression. The elastic–viscoplastic behavior is described by two constitutive equations

- the elastic part relates the elastic strain rate $\dot{\varepsilon}^{e}$ to the stress rate $\dot{\sigma}$ through the elastic moduli c,
- the viscoplastic strain rate $\dot{\varepsilon}^{vp}$ is related to stress σ, viscoplastic strain ε^{vp}, temperature...

The global behavior of a RVE is directly determined by averaging the local fields, the difficulty being mainly in the account of the elastic–viscoplastic nature of the interactions between them. The first models (Weng, 1981), (Nemat-Nasser and Obata, 1986), deduced directly from the stiff interaction law of Kröner (1961), overestimate the internal stresses and are not far from Taylor (1938) and Lin (1957) uniform fields approximations. To overcome these restrictions and build an elastic–viscoplastic rule for the interactions between the heterogeneities, a special relation $(C : M = \lambda I)$ was required between the overall elastic moduli C and the viscoplastic compliance M in the model of Kouddane et al. (1993). Later, Li and Weng (1994) have modified their first model introducing softer time-dependent interaction between the different parts of the material. In the work of Toth and Molinari (1994), a scalar interaction parameter is introduced by tuning the self-consistent predictions with finite element results.

The tangent linearization introduced by Rougier et al. (1994) was also extended to an affine self-consistent model by Masson and Zaoui (1999).

A complete mechanical formulation based on translated fields, projection operators and self-consistent approximation of integral equations (Paquin et al., 1999), was proposed.

Its main characteristics is that the symmetry between elasticity and viscoplasticity is preserved using a projection operator formed by the sum of the operators associated with the elastic and the viscoplastic problems. A complex interaction formula is deduced from translated fields with respect to those calculated

in the case of pure elastic and pure viscous situations. The resulting model was successfully applied to the determination of the overall behavior of composites as well as polycrystalline heterogeneous materials (Paquin et al., 2001).

The new model presented in this paper rather follows Kröner–Weng's scheme than Paquin et al.'s process: the elastic moduli are written as fluctuations regarding elastic reference moduli and the viscoplastic strain rate is written as fluctuations regarding a reference strain rate that has to be compatible but not necessary uniform as it has to be in the Kröner–Weng's model. Thus, the present model does not try to respect the symmetry between elasticity and viscoplasticity as Paquin et al.'s model does. Projection operator and translated field techniques are simultaneously used whereas, in the case of Paquin et al.'s model, at first, a special projection operator is applied to elastic and viscoplastic fields and then, the elastic and viscoplastic strain rates are translated to perform the self-consistent approximation.

In the first part of this work, the background of micro-mechanical tools are recalled especially the projection operators, their properties and the self-consistency condition leading to reduce the interactions between the "particles" to the local ones. For simplicity, this is done in the case of linear elasticity. The second part deals with elastic–viscoplastic heterogeneous materials having a Maxwellian local behavior. The elasticity of the medium is determined by the usual self-consistent approach while, for the viscoplastic interactions, translated fields are introduced. At the same time, Kröner's model is reconsidered in the framework of the translated field technique. The model is first applied to the case of two-phase composites with isotropic, linear and incompressible viscoelastic properties for which an exact self-consistent solution (using the Laplace–Carson transform) is available (Rougier et al., 1994). The agreement between both models appears quite good. Results for elastic–viscoplastic BCC polycrystals are also presented and compared with those obtained from Kröner–Weng's and Paquin et al.'s models.

2. Integral equation, projection operators and self-consistent approximation

This part recalls the method providing, from the field equations and the local behavior (here linear elastic), the integral equations established by Dederichs and Zeller (1973) and Kröner (1967) with the help of the Green function of the problem. At the same time, the notations used throughout this paper are introduced. Thanks to the application of projection operators owning noticable properties, mainly developed by Kunin (1983) and Kröner (1986), a new formulation of the integral equation is written. This one is more appropriate for the introduction of translated fields, very powerful in the scale transition theories. Finally, the self-consistency condition, proposed by Kröner (1986) using the properties of decomposition of the Green functions into local and non-local parts is detailed.

2.1. Formulation in the case of heterogeneous linear elasticity

Let V be an elastic heterogeneous medium with local elastic moduli $c(r)$ (elastic compliances $s(r) = c^{-1}(r)$). The displacement $u^{\mathrm{d}} = E \cdot x$ over ∂V is prescribed at the boundary ∂V (E is uniform over ∂V). Through the use of the scale transition theories and by averaging the local strain and stress fields ε and σ over the RVE, the macroscopic constitutive equation linking the macroscopic strain E to the macroscopic stress Σ is determined

$$E = \frac{1}{V} \int_V \varepsilon(r)\, \mathrm{d}V = \overline{\varepsilon}, \qquad \Sigma = \frac{1}{V} \int_V \sigma(r)\, \mathrm{d}V = \overline{\sigma}. \tag{1}$$

The problem consists in finding displacement, strain and stress fields, $u(r)$, $\varepsilon(r)$ and $\sigma(r)$ satisfying field equations in linear elasticity

$$\sigma = c : \varepsilon,$$
$$\operatorname{div} \sigma = 0,$$
$$\varepsilon = \nabla^{s} u,$$
$$u = E \cdot x \quad \text{on } \partial V. \tag{2}$$

The notation $\nabla^{s} v$ indicates the symmetrical part of the velocity gradient. The symbol ":" is used for the contracted product between two tensors and "$\cdot$" indicates the scalar product between two vectors.

The local elastic moduli $c(r)$ are decomposed into a uniform part C (corresponding to the elastic moduli of an homogeneous reference medium) and a fluctuating part $\delta c(r)$

$$c(r) = C + \delta c(r). \tag{3}$$

The Navier equation can thus be written as

$$\operatorname{div}(C : \nabla^{s} u) + \operatorname{div}(\delta c : \nabla^{s} u) = 0. \tag{4}$$

Introducing the Green functions $G^{C}(r)$ of the elastic reference medium such as

$$C_{ijkl} G^{C}_{km,lj}(r) + \delta_{im} \delta(r) = 0,$$
$$G^{C}(r) = 0 \quad \text{on } \partial V, \tag{5}$$

where $\delta(r)$ represents the Dirac function and δ_{im} is the Kronecker symbol, the set of partial derivative equations (4) becomes an integral equation (Dederichs and Zeller, 1973; Kröner, 1967)

$$\varepsilon(r) = E - \int_{V} \Gamma^{C}(r - r') : \delta c(r') : \varepsilon(r') \, dV.$$

By denoting $*$ the space convolution product, it can be recast

$$\varepsilon = E - \Gamma^{C} * \delta c : \varepsilon. \tag{6}$$

Γ^{C} is the modified Green tensor associated to the reference medium and defined by

$$\Gamma^{C}_{ijkl} = -\tfrac{1}{2} \left(G^{C}_{ik,jl} + G^{C}_{jk,il} \right). \tag{7}$$

This integral equation is simplified in many approximate methods such as the self-consistent model to provide the effective mechanical behavior of heterogeneous materials.

2.2. Projection operator properties

The projection operator Π^{C} related to the modified Green tensor Γ^{C} was first introduced by Kunin (1983)

$$\Pi^{C} = \Gamma^{C} : C. \tag{8}$$

Some of its properties are useful when compatible or balanced fields are introduced. Thus, any balanced stress field σ displays the following property

$$\operatorname{div} \sigma = 0 \Longleftrightarrow \Pi^{C} * S : \sigma = 0 \tag{9}$$

with $S = C^{-1}$. In a similar way, any compatible strain field ε satisfies

$$\varepsilon = \nabla^{s} u, \qquad u^{d} = E \cdot x \Longleftrightarrow \Pi^{C} * \varepsilon = \varepsilon - E. \tag{10}$$

The demonstrations of (9) and (10) are recalled in Fourier space where space convolution products become simple products. Let the vector $k(k_{1}, k_{2}, k_{3})$ be the conjugate of the vector $r(x_{1}, x_{2}, x_{3})$ in Fourier space. The Fourier transform of the Green function $G^{C}_{jn}(r)$ is defined by

$$\widetilde{G}^C_{jn}(k) = \int_V G^C_{jn}(r)\mathrm{e}^{-\mathrm{i}k.r}\,\mathrm{d}V \tag{11}$$

with the following usual properties

$$\widetilde{G}^C_{jn,i} = \mathrm{i}k_i\widetilde{G}^C_{jn}, \qquad \widetilde{G}^C_{jn,ik} = -k_ik_k\widetilde{G}^C_{jn}.$$

The Navier equation (5) defining G^C is recast as

$$C_{ijkl}\widetilde{G}^C_{jn}k_ik_k = \delta_{ln}. \tag{12}$$

In Fourier space, (9) is equivalent to

$$\tilde{\sigma}_{ij}k_j = 0 \iff \widetilde{\Pi}^C_{mnpq}S_{pqij}\tilde{\sigma}_{ij} = 0.$$

Assuming that $\tilde{\sigma}_{ij}k_j = 0$, therefore

$$\widetilde{\Pi}^C_{mnpq}S_{pqij}\tilde{\sigma}_{ij} = \widetilde{\Gamma}^C_{mnij}\tilde{\sigma}_{ij} = \tfrac{1}{2}\Big(\widetilde{G}^C_{jm}k_ik_n + \widetilde{G}^C_{jn}k_ik_m\Big)\tilde{\sigma}_{ij} = \tfrac{1}{2}\Big(\widetilde{G}^C_{jm}k_n + \widetilde{G}^C_{jn}k_m\Big)\tilde{\sigma}_{ij}k_i = 0.$$

Inversely, if: $\widetilde{\Pi}^C_{mnpq}S_{pqij}\tilde{\sigma}_{ij} = 0$, then, one can easily deduced that $(\widetilde{G}^C_{jm}k_n + \widetilde{G}^C_{jn}k_m)\tilde{\sigma}_{ij}k_i = 0$, and necessarily $\tilde{\sigma}_{ij}k_i = 0$.

In the case of compatible strain fields

$$\varepsilon = \nabla^s u \quad \text{and} \quad u^\mathrm{d} = E \cdot x \quad \text{on } \partial V$$

the following decomposition

$$\varepsilon = E + \varepsilon' \quad \text{with} \quad \varepsilon' = \nabla^s u'$$

is introduced.

Using the property $\Pi^C * E = 0$, one obtains $\Pi^C * \varepsilon = \Pi^C * \varepsilon'$ that leads to

$$\widetilde{\Pi}^C_{mnij}\tilde{\varepsilon}'_{ij} = \Gamma^C_{mnij}C_{ijkl}\tilde{u}'_{k,l} = \tfrac{1}{2}\Big(\widetilde{G}^C_{jm}k_nk_i + \widetilde{G}^C_{jn}k_ik_m\Big)C_{ijkl}\tilde{u}'_l\mathrm{i}k_k.$$

Taking into account (12), one obtains

$$\widetilde{\Pi}^C_{mnij}\tilde{\varepsilon}'_{ij} = \tfrac{1}{2}\Big(C_{ijkl}\widetilde{G}^C_{jn}k_kk_i\mathrm{i}k_m + C_{ijkl}\widetilde{G}^C_{jm}k_ik_m\mathrm{i}k_n\Big)\tilde{u}'_1 = \tfrac{1}{2}(\delta_{ln}\mathrm{i}k_m + \delta_{lm}\mathrm{i}k_n)\tilde{u}'_1 = \tfrac{1}{2}\Big(\tilde{u}'_{n,m} + \tilde{u}'_{m,n}\Big) = \tilde{\varepsilon}'_{mn}$$

$$= \tilde{\varepsilon}_{mn} - \widetilde{E}_{mn}$$

and finally $\Pi^C * \varepsilon' = \Pi^C * \varepsilon = \varepsilon - E$.

Inversely, if $\Pi^C * \varepsilon = \nabla^s u - E$, applying this equality at the boundary ∂V, one deduces directly $u_i = E_{ij}x_j$ on ∂V. Introducing $u_i = u_i^\mathrm{d} + u_i'$, with $u_i^\mathrm{d} = E_{ij}x_j$, it comes $\nabla^s u - E = \nabla^s u'$.

Now

$$\widetilde{\Pi}^C_{mnij}\tilde{\varepsilon}'_{ij} = \tilde{u}'_{n,m} = \delta_{ln}\tilde{u}'_{l,m} = C_{ijkl}\widetilde{G}^C_{jn}k_kk_i\tilde{u}'_l\mathrm{i}k_m = -C_{ijkl}\widetilde{G}^C_{jn,im}\tilde{u}'_{l,k}$$

so that

$$\tfrac{1}{2}\Big(\tilde{u}'_{n,m} + \tilde{u}'_{m,n}\Big) = -\tfrac{1}{2}\Big(\widetilde{G}^C_{jn,im} + \widetilde{G}^C_{jm,in}\Big)C_{ijkl}\tilde{u}'_{l,k} = \widetilde{\Gamma}^C_{mnij}C_{ijkl}\tilde{u}'_{l,k}.$$

With $\nabla^s u' = \Pi^C * \nabla^s u'$

one gets finally

$$\nabla^s u - E = \Pi^C * \nabla^s u = \Pi^C * \varepsilon.$$

2.3. Reformulation of the elastic heterogeneous problem

The interest of the projection operator Π^C is to provide an adimensional representation of Green functions Γ^C. Using the properties (9) and (10), it allows to transform the problem (2) and (3) into the following set of equations, that is homogeneous in deformation

$$
\begin{aligned}
S : \sigma &= \varepsilon - \delta s : c : \varepsilon, \\
\Pi^C * S : \sigma &= 0, \\
\Pi^C * \varepsilon &= \varepsilon - E,
\end{aligned}
\tag{13}
$$

with $\delta s = s - S$.

Condition (13) (second term) is equivalent to equilibrium equations (2) (second term). Moreover, only the condition (13) (third term) concerning the strain field is required to fulfill compatibility conditions (2) (third and fourth terms).

Applying the projection operator Π^C to the first term of Eq. (13) and taking into account the properties ((9) and (10)), one directly gets the integral equation

$$
\varepsilon = E + \Pi^C * \delta s : c : \varepsilon.
\tag{14}
$$

This one is based on projection operators amounts to the classical integral equation (6).

2.4. Self-consistent approximation

The analytical solution of the integral equation is usually neither possible nor required to determine the effective properties of a (linear) heterogeneous medium. In many cases, approximations are based on familiar solutions of elementary problems of inclusion(s) embedded in an infinite matrix or on hypothesis concerning the whole fields. The self-consistent approximation, introduced by Kröner (1958) is based on the decomposition of the modified Green function Γ^C or the projection operator Π^C into local (l) part and non-local (nl) part such as

$$
\begin{aligned}
\Gamma^C(r - r') &= \Gamma_l^C \delta(r - r') + \Gamma_{nl}^C(r - r'), \\
\Pi^C(r - r') &= \Pi_l^C \delta(r - r') + \Pi_{nl}^C(r - r').
\end{aligned}
\tag{15}
$$

Replacing (15) in the integral Eq. (14), it follows:

$$
\varepsilon = E + \Pi_l^C : \delta s : c : \varepsilon + \Pi_{nl}^C * \delta s : c : \varepsilon.
\tag{16}
$$

The second right term expresses the local effect of the elastic heterogeneity on the strain field ε. On the other hand, the third term that contains the convolution involves the long distance interactions. Because of their complexity, their description is usually harder.

By applying the self-consistent scheme, this third term is not exactly evaluated but its contribution is reduced by imposing on the average of the quantity $\delta s : c : \varepsilon$ over the RVE to vanish

$$
\overline{\delta s : c : \varepsilon} = 0.
\tag{17}
$$

Taking into account (1) and (2), (17) provides the macroscopic constitutive equation $\Sigma = C : E$. Because of the condition (17), the reference medium is required to be the effective equivalent medium: $C = C^e$. The non-local contribution of (16) is then neglected and the following concentration relations are deduced

$$
\varepsilon = A^{C^e} : E, \qquad \sigma = c : A^{C^e} : E,
\tag{18}
$$

where A^{C^e} represents the strain concentration tensor defined by

$$A^{C^e} = \left(I + \Gamma_1^{C^e} : \delta c^e\right)^{-1}, \qquad \delta c^e = c - C^e. \tag{19}$$

The effective elastic tensor C^e are determined from the following equations:

$$C^e = \overline{c : A^{C^e}}, \qquad \overline{A^{C^e}} = I. \tag{20}$$

$\Gamma_1^{C^e}$ is the local part of the modified Green tensor Γ^{C^e} associated to the effective medium C^e and I is the fourth-order identity tensor.

The same results could have been obtained for a heterogeneous viscoplastic solid, which will be treated in the following part.

The preceding results are now used to build an integral equation for the elastic–viscoplastic material and to apply the self-consistent approximation with the help of the projection operators properties for elasticity.

3. New model for elastic–viscoplastic heterogeneous materials

This part aims to express the relation between local fields and boundary conditions in the case of elastic–viscoplastic heterogeneous medium using results established in the preceding section. Let consider an elastic heterogeneous medium V, with local elastic moduli $c(r)$ (elastic compliances $s(r) = c^{-1}(r)$) and secant viscoplastic compliances $m(r)$ (secant moduli $b(r)$). The velocity $v^d = \dot{E} \cdot x$, where $\dot{E}$ is a uniform strain rate, is prescribed at the boundary ∂V. The problem is treated in the framework of quasi-static equilibrium with no volume forces. In the non-linear case, $m(r)$ depends on stress σ and viscoplastic strain $\varepsilon^{vp}(r)$. The equations of the elastic–viscoplastic heterogeneous problem are thus given by

- constitutive equation

$$\dot{\varepsilon}(r, t) = s(r) : \dot{\sigma}(r, t) + \dot{\varepsilon}^{vp}(r, t), \tag{21}$$

- evolution law of the viscoplastic strain

$$\dot{\varepsilon}^{vp}(r, t) = m(\sigma, \varepsilon^{vp}, r) : \sigma(r, t), \tag{22}$$

- equilibrium equation for the unknown field $\dot{\sigma}$

$$\operatorname{div} \dot{\sigma} = 0, \tag{23}$$

- kinematical compatibility relation

$$\dot{\varepsilon} = \nabla^s v, \tag{24}$$

- boundary conditions

$$v_i^d = \dot{E}_{ij} x_j \quad \text{on } \partial V. \tag{25}$$

In addition, the stress field σ has to check the property $\operatorname{div} \sigma = 0$.

The problem consists in finding stress and strain rates fields $\dot{\sigma}$ and $\dot{\varepsilon}$ satisfying field equations (21)–(25) that allow to determine the overall properties of the RVE through the homogenization step.

An elastic homogeneous medium C and its associated projection operator Π^C are introduced. By applying Π^C and using the properties (9) and (10), the equations of the present problem are transformed into the following ones:

$$S : \dot{\sigma} = \dot{\varepsilon} - \delta s : c : \dot{\varepsilon}^e - \dot{\varepsilon}^{vp}, \tag{26}$$

$$\Pi^C * S : \dot{\sigma} = 0, \tag{27}$$

$$\Pi^C * \dot{\varepsilon} = \dot{\varepsilon} - \dot{E}. \tag{28}$$

The Eqs. (26)–(28) have to be simultaneously satisfied. Hence, the operator Π^C is applied to the Eq. (26), so that

$$\Pi^C \ast S : \dot{\sigma} = \Pi^C \ast (\dot{\varepsilon} - \delta s : c : \dot{\varepsilon}^{\mathrm{e}} - \dot{\varepsilon}^{\mathrm{vp}})$$

and the properties (27) and (28) give rise to

$$\dot{\varepsilon} = \dot{E} + \Pi^C \ast (\delta s : c : \dot{\varepsilon}^{\mathrm{e}} + \dot{\varepsilon}^{\mathrm{vp}}). \tag{29}$$

The integral equation (29) reflects the interactions between the elastic and viscoplastic heterogeneities in the material: the strain rate $\dot{\varepsilon}(r)$ at point r depends on the macroscopic loading $\dot{E}$, on the elastic heterogeneities $\delta c(r)$ and on the viscoplastic strain rate field $\dot{\varepsilon}^{\mathrm{vp}}(r)$ in the whole volume V.

This integral equation is equivalent to the classical one developed by Kröner (1977) and Berveiller et al. (1987) with a different formulation based on the modified Green tensor Γ^C

$$\dot{\varepsilon} = \dot{E} - \Gamma^C \ast (\delta c : \dot{\varepsilon} : -c : \dot{\varepsilon}^{\mathrm{vp}}). \tag{30}$$

The integral equations (29) and (30) represent completely the mechanical problem of heterogeneous elastoviscoplasticity so long as the viscoplastic behavior, as it is specified by (22), is taken into account. Now, this is not done in the current forms (29) and (30) of the integral equation. Because of the viscoplastic strain rate independency of the effective elastic properties (moduli C^{e}) of the equivalent elastic–viscoplastic medium, the self-consistent approximation, recalled in Section 2.4, can be directly used to definitely determine the elastic moduli C introduced in (26).

Taking $C = C^{\mathrm{e}}$ allows to satisfy the self-consistency conditions for the elastic part and the integral equation (29) becomes

$$\dot{\varepsilon} = \dot{E} + \Pi^{C^{\mathrm{e}}} \ast (\delta s^{\mathrm{e}} : c : \dot{\varepsilon}^{\mathrm{e}} + \dot{\varepsilon}^{\mathrm{vp}}), \tag{31}$$

where $\delta s^{\mathrm{e}} = s - S^{\mathrm{e}}$ with $S^{\mathrm{e}} = C^{\mathrm{e}-1}$, C^{e} and $\Pi^{C^{\mathrm{e}}}$ being evaluated from the Eqs. (8) and (20). The choice of the effective elastic moduli, according to the self-consistent approximation, is not sufficient to completely solve the integral equation (31). The reason is twofold: on the one hand, (31) is still a spatial convolution and on the other hand, the viscoplastic behavior $\dot{\varepsilon}^{\mathrm{vp}} = m : \sigma$ is not yet introduced in the interaction law (31). Only using the vanishing of the average of $(\delta s^{\mathrm{e}} : c : \dot{\varepsilon}^{\mathrm{e}} + \dot{\varepsilon}^{\mathrm{vp}})$, to reduce the right-hand side of (31) to its local part, leads to the already introduced definition of the mean values of $\dot{\Sigma}$ at $\dot{E}$ (pure elasticity). To reduce (31) to its local part, the method developed by Kröner (1961) in elastoplasticity or by Weng (1981) in elastoviscoplasticity is firstly followed. While not giving an explicit local plastic or viscoplastic law, the preceding authors have implicitly used the possibility offered by the translated fields coming from the properties of the operators Π^C.

3.1. Kröner–Weng's classical self-consistent approximation

This approximation consists in introducing fluctuations of the viscoplastic strain rate field $\dot{\varepsilon}^{\mathrm{vp}}$ regarding a uniform viscoplastic strain rate $\dot{E}^{\mathrm{vp}}$ (not necessary the average of $\dot{\varepsilon}^{\mathrm{vp}}$) such as

$$\dot{\varepsilon}^{\mathrm{vp}}(r) = \dot{E}^{\mathrm{vp}} + \delta\dot{\varepsilon}^{\mathrm{vp}}(r). \tag{32}$$

At this step, the viscoplastic law $\dot{\varepsilon}^{\mathrm{vp}} = m : \sigma$ is not introduced yet.

The projection operator $\Pi^{C^{\mathrm{e}}}$ calculated from the effective elastic moduli C^{e} is applied to the uniform field $\dot{E}^{\mathrm{vp}}$, so that, from the properties (10), one gets

$$\Pi^{C^{\mathrm{e}}} \ast \dot{E}^{\mathrm{vp}} = 0. \tag{33}$$

The integral equation (31) becomes

$$\dot{\varepsilon} = \dot{E} + \Pi^{C^{\mathrm{e}}} * (\delta s^{\mathrm{e}} : c : \dot{\varepsilon}^{\mathrm{e}} + \delta \dot{\varepsilon}^{\mathrm{vp}}). \tag{34}$$

In the case of Kröner's self-consistent approximation, $\dot{E}^{\mathrm{vp}}$ is chosen in order to weaken the contribution of the non-local integral term by vanishing the average of $(\delta s^{\mathrm{e}} : c : \dot{\varepsilon}^{\mathrm{e}} + \dot{\varepsilon}^{\mathrm{vp}})$ such as

$$\overline{\delta s^{\mathrm{e}} : c : \dot{\varepsilon}^{\mathrm{e}} + \delta \dot{\varepsilon}^{\mathrm{vp}}} = 0. \tag{35}$$

This condition imposed the value of the uniform viscoplastic field $\dot{E}^{\mathrm{vp}}$ as being the one of the effective viscoplastic strain rate $\dot{E}^{\mathrm{vp}} = \dot{E}^{\mathrm{vpe}}$

$$\dot{E} = S^{\mathrm{e}} : \dot{\Sigma} + \dot{E}^{\mathrm{vpe}}. \tag{36}$$

Only keeping the local part of (34) gives

$$\dot{\varepsilon} = A^{C^{\mathrm{e}}} : \dot{E} + A^{C^{\mathrm{e}}} : \Gamma_1^{C^{\mathrm{e}}} : (c : \dot{\varepsilon}^{\mathrm{vp}} - C^{\mathrm{e}} : \dot{E}^{\mathrm{vpe}}) \tag{37}$$

with

$$\dot{E}^{\mathrm{vpe}} = \overline{{}^t B^{C^{\mathrm{e}}} : \dot{\varepsilon}^{\mathrm{vp}}}, \tag{38}$$

where $B^{C^{\mathrm{e}}}$ denotes the elastic stress concentration tensor

$$B^{C^{\mathrm{e}}} = c : A^{C^{\mathrm{e}}} : S^{\mathrm{e}} \quad \text{with } {}^t B_{ijkl}^{C^{\mathrm{e}}} = B_{klij}^{C^{\mathrm{e}}}. \tag{39}$$

In the case of homogeneous elasticity ($c = C^{\mathrm{e}}$), the well-known Kröner's (1961) relation is thus re-established. This one has been used later by many authors with the following strain rate formulation

$$\dot{\sigma} = \dot{\Sigma} + C^{\mathrm{e}} : (S^{\mathrm{E}} - I) : (\dot{\varepsilon}^{\mathrm{vp}} - \dot{E}^{\mathrm{vpe}}) \tag{40}$$

with $\dot{E}^{\mathrm{vpe}} = \overline{\dot{\varepsilon}^{\mathrm{vp}}}$ and $S^{\mathrm{E}} = \Gamma_1^{C^{\mathrm{e}}} : C^{\mathrm{e}}$ is the well-known Eshelby (1957) tensor. It should be underlined that this relation is exact as far as mathematical and physical aspects are concerned. However, the mechanical interactions remain roughly estimated. As a matter of fact, the inelastic behavior is not taken into account in this formulation. This means that the inelastic strains actually depend on the stresses and therefore are not "stress free strains" as they are considered in Kröner's model. The formulation based on (37) or (40) strongly overestimates the internal stresses (Berveiller and Zaoui, 1979) and leads to interactions close to those contained by the Taylor (1938) and Lin (1957) models.

As it is showed, the preceding method displays forces and weaknesses. Its analyze leads to propose a new class of models based on the introduction of translated fields, more realistic than the preceding ones, in the integral equation (31).

3.2. New self-consistent approximation based on translated fields

The classical self-consistent approximation (Section 2.4) is well-suited for classical bi-univocal laws (elasticity, elastoplasticity, viscoplasticity) but appears not appropriate for differential constitutive equations which involve different order of time derivation concerning stress and/or strain fields like elastic–viscoplastic materials.

Hence, in an alternative approach, a less conventional self-consistent approximation based on projection operator Π^C properties defined by the Eqs. (9) and (10) is proposed. Thus, these specific properties lead to apply self-consistency on particular translated fields (that fulfill the kinematical compatibility or equilibrium conditions) that respect these properties. In the present case, translating the field $\dot{\varepsilon}^{\mathrm{vp}}(r)$ with regarding a non-necessarily uniform compatible one $\dot{e}^{\mathrm{vp}}(r)$ gives

$$\dot{\varepsilon}^{\mathrm{vp}}(r) = \dot{e}^{\mathrm{vp}}(r) + \delta \dot{\varepsilon}^{\mathrm{vp}}(r).$$

The elastic–viscoplastic heterogeneous material can be schematically represented by an assembling of elastic and viscoplastic elements. As a result, it is characterized by its properties corresponding to the cases of pure elasticity and pure viscoplasticity. These properties can be associated with particular mechanical states that physically correspond to the asymptotic situations reached by the material.

The pure elastic state is described in the Section 2.4. The treatment used is simple and independent on the pure viscoplastic state. Similarly, the case of pure viscoplasticity and the corresponding integral equation of a heterogeneous viscoplastic material is solved by the self-consistent approximation. This provides a compatible viscoplastic strain rate field useful to carry out the required translation in the resolution of the elastic–viscoplastic heterogeneous problem.

3.2.1. Case of pure viscoplasticity

In that case, the constitutive equation (21) is reduced to

$$\dot{\varepsilon} = \dot{\varepsilon}^{\mathrm{vp}} = m : \sigma. \tag{41}$$

As done for elastic properties, a reference medium having viscoplastic homogeneous compliances M (secant moduli B) is introduced such as

$$m(r) = M + \delta m(r), \qquad b(r) = B + \delta b(r). \tag{42}$$

Owing to the local viscoplastic law (41), to the kinematical compatibility and boundary conditions (24) and (25), to the expressions (42) and moreover to the equilibrium property of the stress field $\mathrm{div}\,\sigma = 0$, one obtains the following integral equation:

$$\dot{\varepsilon} = \dot{E} - \Gamma^B * \delta b : \dot{\varepsilon}. \tag{43}$$

Γ^B denotes the modified Green tensor associated to the viscoplastic reference medium B.

The integral equation (43) is similar to the one of the elastic heterogeneous problem (6). The self-consistent approximation of (43) provides the strain rate concentration relation

$$\dot{\varepsilon} = A^{B^e} : \dot{E}, \tag{44}$$

where A^{B^e} is the concentration tensor of the viscoplastic strain rate associated with the effective viscoplastic moduli B^e defined by

$$A^{B^e} = \left(I + \Gamma_1^{B^e} : \delta b^e \right)^{-1}, \qquad \delta b^e = b - B^e. \tag{45}$$

The effective viscoplastic moduli B^e are deduced from the following equations:

$$B^e = \overline{b : A^{B^e}}, \qquad \overline{A^{B^e}} = I. \tag{46}$$

The prediction of the compatible field $\dot{\varepsilon}$ defined by (44) is exact in the case of a complete resolution of the integral equation (43) or approximated in the case of models such as those developed by Taylor (1938), Mori and Tanaka (1973) or the viscoplastic self-consistent scheme.

3.2.2. Case of elastoviscoplasticity

In the present case, the viscoplastic self-consistent approximation is used to simplify the integral equation (43) and to provide a compatible, not necessarily uniform, $\dot{e}^{\mathrm{vp}}(r)$ required for the translation of the elastic–viscoplastic problem. Thus, any field $\dot{e}^{\mathrm{vp}}(r)$ having the form

$$\dot{e}^{\mathrm{vp}}(r) = A^{B^e}(r) : \dot{X} \tag{47}$$

has the following property

$$\Pi^{C^e} * A^{B^e}(r) : \dot{X} = \dot{e}^{\mathrm{vp}}(r) - \dot{X}, \tag{48}$$

where $\dot{X}$ denotes any tensor chosen to define the kinematical compatible fields $\dot{e}^{\mathrm{vp}}(r)$. The fluctuation of the viscoplastic strain rate field $\dot{\varepsilon}^{\mathrm{vp}}$ regarding the field $\dot{e}^{\mathrm{vp}}(r) = A^{B^e}(r) : \dot{X}$ is defined by taking into account the property (48) such as

$$\dot{\varepsilon}^{\mathrm{vp}}(r) = A^{B^e}(r) : \dot{X} + \delta\dot{\varepsilon}^{\mathrm{vp}}(r). \tag{49}$$

This decomposition and the property (48) lead to the following integral equation

$$\dot{\varepsilon} = \dot{E} + A^{B^e} : \dot{X} - \dot{X} + \Pi^{C^e} \ast (\delta s^e : c : \dot{\varepsilon}^e + \delta\dot{\varepsilon}^{\mathrm{vp}}). \tag{50}$$

According to the self-consistency condition, the unknown tensor $\dot{X}$ is chosen so that the average of $\delta s^e : c : \dot{\varepsilon}^e + \delta\dot{\varepsilon}^{\mathrm{vp}}$ vanishes

$$\overline{\delta s^e : c : \dot{\varepsilon}^e + \delta\dot{\varepsilon}^{\mathrm{vp}}} = 0. \tag{51}$$

Owing to the preceding equations and the property (46), $\dot{X}$ is deduced from (51)

$$\dot{X} = \overline{\delta s : \dot{\sigma} + \dot{\varepsilon}^{\mathrm{vp}}} = \dot{E} - S^e : \dot{\Sigma}. \tag{52}$$

$\dot{X}$ can be re-arranged as the following form obtained after identification of the macroscopic constitutive equations (36) and (38)

$$\dot{X} = \dot{E}^{\mathrm{vpe}} = \overline{{}^t B^{C^e} : \dot{\varepsilon}^{\mathrm{vp}}}. \tag{53}$$

Substituting (53) in (50), and neglecting the non-local term produce

$$\dot{\varepsilon} = \dot{E} + A^{B^e} : \dot{E}^{\mathrm{vpe}} - \dot{E}^{\mathrm{vpe}} + \Pi_1^{C^e} : (\delta s^e : c : \dot{\varepsilon}^e + \delta\dot{\varepsilon}^{\mathrm{vp}}). \tag{54}$$

Combining the preceding Eqs. (19), (21), (49) and (54) finally gives

$$\dot{\varepsilon} = A^{C^e} : (\dot{E} - \dot{E}^{\mathrm{vpe}}) + A^{C^e} : A^{B^e} : \dot{E}^{\mathrm{vpe}} + A^{C^e} : \Gamma_1^{C^e} : (c : \dot{\varepsilon}^{\mathrm{vp}} - C^e : A^{B^e} : \dot{E}^{\mathrm{vpe}}), \tag{55}$$

where $\dot{\varepsilon}^{\mathrm{vp}}(r,t) = m(\sigma, \varepsilon^{\mathrm{vp}}, r) : \sigma(r,t)$ and $\dot{E}^{\mathrm{vpe}} = \overline{{}^t B^{C^e} : \dot{\varepsilon}^{\mathrm{vp}}}$. Substituting the strain rate concentration relation (55) in the local constitutive equation (21) gives the following interaction law for stresses

$$\dot{\sigma} = c : A^{C^e} : S^e : \dot{\Sigma} + c : A^{C^e} : (S^E - I) : (\dot{\varepsilon}^{\mathrm{vp}} - A^{B^e} : \dot{E}^{\mathrm{vpe}}). \tag{56}$$

This equation contains two asymptotic states

pure elasticity: $\dot{\varepsilon}^{\mathrm{vp}} = \dot{E}^{\mathrm{vpe}} = 0$

$$\dot{\sigma} = c : A^{C^e} : \dot{E}, \tag{57}$$

pure viscoplasticity: $\dot{\sigma} = \dot{\Sigma} = 0$

$$\sigma = b : A^{B^e} : \dot{E}. \tag{58}$$

In the case of an elastic homogeneous behavior ($c = C^e$, $A^{C^e} = I$), (55) becomes

$$\dot{\varepsilon} = \dot{E} - \dot{E}^{\mathrm{vp}} + A^{B^e} : \dot{E}^{\mathrm{vp}} + S^E : (\dot{\varepsilon}^{\mathrm{vp}} - A^{B^e} : \dot{E}^{\mathrm{vp}}) \tag{59}$$

and (56) is replaced by

$$\dot{\sigma} = \dot{\Sigma} + C^e : (S^E - I) : (\dot{\varepsilon}^p - A^{B^e} : \dot{E}^{\mathrm{vp}}) \tag{60}$$

with $\dot{E}^{\mathrm{vp}} = \overline{\dot{\varepsilon}^{\mathrm{vp}}}$.

Unlike Kröner's interaction law (37), that only contains the asymptotic state of pure elasticity, (56) expresses in addition the asymptotic state of pure viscoplasticity.

3.3. Isotropic behaviors and spherical inclusions

Eqs. (55) and (56) are valid for a RVE constituted of spherical inclusions. In this part, they are simplified for the case of isotropic behavior.

Let us suppose that the inclusions are spheres with an isotropic behavior and distributed randomly in the matrix so that the overall behavior is also isotropic. This case simply and directly derives from Eq. (55) or (56). The expressions of the concentration relations are thus explicit.

The isotropic behavior of each phase and of the overall material is defined by Lamé moduli (λ and μ for elasticity, α and η for viscoplasticity). The elastic c, C and viscoplastic b, B tensors are defined by

$$c^I_{ijkl} = 2\mu^I I_{ijkl} + \lambda^I \delta_{ij}\delta_{kl} \quad \text{and} \quad b^I_{ijkl} = 2\eta^I I_{ijkl} + \alpha^I \delta_{ij}\delta_{kl}$$

for the inclusion (*I*) and by

$$C_{ijkl} = 2\mu I_{ijkl} + \lambda \delta_{ij}\delta_{kl} \quad \text{and} \quad B_{ijkl} = 2\eta I_{ijkl} + \alpha \delta_{ij}\delta_{kl}$$

for the matrix.

The moduli λ and α are related to Poisson ratios and shear moduli such as

$$\lambda^I = \frac{2\mu^I v^I}{1 - 2v^I}, \quad \alpha^I = \frac{2\eta^I v^{I^V}}{1 - 2v^{I^V}}, \quad \lambda = \frac{2\mu v}{1 - 2v}, \quad \alpha = \frac{2\eta v^v}{1 - 2v^v}.$$

The tensor T^I for an isotropic medium is defined by the shear modulus and the Poisson ratio of the matrix (μ et v for elasticity, η et v^v for the viscosity) (Kröner, 1989) by

$$T^{C^I}_{ijkl} = \frac{4 - 5v}{15\mu(1 - v)} I_{ijkl} - \frac{1}{30\mu(1 - v)} \delta_{ij}\delta_{kl},$$

$$T^{B^I}_{ijkl} = \frac{4 - 5v^v}{15\eta(1 - v^v)} I_{ijkl} - \frac{1}{30\eta(1 - v^v)} \delta_{ij}\delta_{kl}.$$

In the case of an incompressible material: $v = 0.5$, $v^v = 0.5$, $\varepsilon_{kk} = 0$, constitutive equations reduce to the deviatoric contribution (suffix D) of the stress and strain tensors

$$\dot{\sigma}^I_D = 2\mu^I \left(\dot{\varepsilon}^I_D - \frac{1}{2\eta^I} \sigma^I_D \right) \tag{61}$$

and

$$A^{C^I} = \frac{5\mu}{3\mu + 2\mu^I}, \qquad A^{B^I} = \frac{5\eta}{3\eta + 2\eta^I}.$$

The interaction law (56) becomes

$$\dot{\sigma}^I = \frac{5\mu^I}{3\mu + 2\mu^I} \left\{ \dot{\Sigma} - 2\mu(1 - \beta) \left(\dot{\varepsilon}^{vpI} - \frac{5\eta}{3\eta + 2\eta^I} \dot{E}^{vp} \right) \right\} \tag{62}$$

with $\beta = \frac{2(4-5v)}{15(1-v)} = \frac{2}{5}$.

In the case of homogeneous elasticity: $\mu^I = \mu$, (62) is written

$$\dot{\sigma}^I = \dot{\Sigma} - 2\mu(1 - \beta) \left(\dot{\varepsilon}^{vpI} - \frac{5\eta}{3\eta + 2\eta^I} \dot{E}^{vp} \right) \tag{63}$$

with $\dot{E}^{vp} = \overline{\dot{\varepsilon}^{vp}}$.

This interaction formula significantly differs from Kröner's equation by the term A^{B^I}.

Originally, Kröner's equation aims to determine the internal stresses related to a plastic incompatibility considered as a "stress free" namely fixed plastic strain. The application of this approach to strain rate fields (Weng's model), and, more generally, of methods of translation around uniform fields, results in evolution equations fundamentally different from (62) and (63). Indeed, the last ones are related to stress-dependent viscoplastic incompatibilities and are new in this sense.

4. Applications to two-phase and polycrystalline materials

4.1. Application to a two-phase incompressible isotropic viscoelastic material

In the case of linear viscoelasticity (constant moduli $c(r)$ and $b(r)$), a comparison of the present model with 1/Rougier et al.'s model (1994) based on the linear viscoelastic inclusion problem solved by Hashin (1969) using the Laplace–Carson transform and 2/Kröner–Weng's model (Section 3) is proposed. The compared results concern the strain rate concentration relation and the macroscopic tensile behavior.

To obtain simple analytical results, an incompressible isotropic composite material constituted of two incompressible isotropic phases is considered. Their elastic and viscous moduli are (μ^1, η^1) and (μ^2, η^2) respectively. The constitutive equation of each phase is given by (61).

The local strain rate $\dot{\varepsilon}^1$ in the phase 1 is calculated by the three models and written in the Laplace–Carson space such as

$$\hat{\dot{\varepsilon}}^{\mathrm{I}} = \widehat{A}(p) : \hat{\dot{E}}. \tag{64}$$

Here, the Laplace–Carson transform of a function f is denoted $\hat{f}$ and defined by

$$\hat{f}(p) = p \int_0^\infty f(t) \mathrm{e}^{-pt} \, \mathrm{d}t.$$

Applying the Laplace–Carson transform to (61) leads to the following constitutive equation for each phase I

$$\hat{s}^I(p) = \frac{2\mu^I}{p + \frac{\mu^I}{\eta^I}} \hat{\dot{e}}^I(p) \quad \text{or} \quad \hat{s}^I(p) = 2\hat{l}^I(p)\hat{\dot{e}}^I(p) \tag{65}$$

with

$$\hat{l}^I(p) = \frac{\mu^I}{p + \frac{\mu^I}{\eta^I}}. \tag{66}$$

4.1.1. Hashin–Rougier's model

The linearity of (65) allows to use the classical self-consistent scheme in the Laplace–Carson space as done by Hashin (1969) and Rougier et al. (1994). Thus, the strain concentration tensor in the Laplace space for the phase I is determined by

$$\widehat{A}^{HI}(p) = \frac{5\widehat{L}^{\mathrm{e}}(p)}{3\widehat{L}^{\mathrm{e}}(p) + 2\hat{l}^I(p)} \tag{67}$$

$\widehat{L}^{\mathrm{e}}$ denotes the effective modulus of the homogeneous equivalent material in the Laplace–Carson space. It is determined by solving the classical homogenization equation of a two-phase material

$$\widehat{L}^{\mathrm{e}} = \hat{l}^2 + f_1(\hat{l}^1 - \hat{l}^2)\widehat{A}^{H^1} \tag{68}$$

that provides

$$\widehat{L}^{\mathrm{e}}(p) = -\frac{2-5f_1}{6}\hat{I}^1(p) - \frac{2-5f_2}{6}\hat{I}^2(p) + \sqrt{\left(\frac{2-5f_2}{6}\hat{I}^1(p) + \frac{2-5f_2}{6}\hat{I}^2(p)\right)^2 + \frac{2}{3}\hat{I}^1(p)\hat{I}^2(p)},\qquad (69)$$

where f_1 and f_2 are respectively the volume fractions of the phases 1 and 2.

4.1.2. Present model

In the case of a two-phase isotropic and incompressible material, the interaction law derived from the present model and (61) is

$$\dot{\sigma}_{\mathrm{D}}^I = a^I\dot{\Sigma} - \frac{6}{5}a^I\mu^{\mathrm{e}}\left(\frac{\sigma_{\mathrm{D}}^I}{2\eta^I} - b^I\overline{a^I\frac{\sigma_{\mathrm{D}}^I}{2\eta^I}}\right)\qquad (70)$$

with $\dot{\Sigma}$ is the macroscopic stress rate. Besides,

$$a^I = \frac{5\mu^I}{3\mu^{\mathrm{e}} + 2\mu^I},\qquad (71)$$

$$b^I = \frac{5\eta^{\mathrm{e}}}{3\eta^{\mathrm{e}} + 2\eta^I}\qquad (72)$$

and μ^{e}, η^{e} are effective elastic and viscous moduli determined by the classical self-consistent scheme for a two-phase material (Berveiller and Zaoui, 1981).

The Laplace–Carson transform of (70) gives for the phase 1

$$\hat{s}^1(p) = \widehat{B}^{\mathrm{PM}^1}(p)\widehat{\Sigma},\qquad (73)$$

where

$$\widehat{B}^{\mathrm{PM}^1}(p) = \frac{p + \frac{6}{5}\frac{\mu^{\mathrm{e}}b^1}{2\eta^2}(1 + a^2 - f_1a^1b^1 - f_1a^2b^2)}{\frac{p}{a^1} + \frac{6}{5}\mu^{\mathrm{e}}\left(\frac{1}{2\eta^1} - \frac{b^1f_1}{2\eta^1}[a^1 + 1 - f_1a^1b^1 - f_2a^2b^2] + \frac{b^1f_1}{2\eta^2}[a^2 + 1 - f_1a^1b^1 - f_2a^2b^2]\right)}.\qquad (74)$$

By using the constitutive equation (61), the following strain concentration relation for the phase 1 is obtained

$$\hat{e}^1(p) = \widehat{A}^{\mathrm{PM}^1}(p)\widehat{E},\qquad (75)$$

where

$$\widehat{A}^{\mathrm{PM}^I}(p) = \frac{\widehat{B}^{\mathrm{PM}^I}(p)\left(p + \frac{\mu^I}{\eta^I}\right)}{2\mu^I\left(a^I\left(\frac{\widehat{B}^{\mathrm{PM}^I}(p)}{2\eta^I} - b^I\overline{\frac{\widehat{B}^{\mathrm{PM}^I}(p)}{2\eta^I}}\right) + \overline{\frac{\widehat{B}^{\mathrm{PM}^I}(p)}{2\eta^I}}\right) + \frac{\mu^I}{\mu^{\mathrm{e}}}p}.\qquad (76)$$

4.1.3. Kröner–Weng's model

In the case of a two-phase isotropic and incompressible material, the interaction law derived from the Kröner–Weng model is

$$\dot{\sigma}_{\mathrm{D}}^I = a^I\dot{\Sigma} - \frac{6}{5}a^I\mu^{\mathrm{e}}\left(\frac{\sigma_{\mathrm{D}}^I}{2\eta^I} - \overline{a^I\frac{\sigma_{\mathrm{D}}^I}{2\eta^I}}\right)\qquad (77)$$

with the same notations as the previous model and with a^I defined as (71).

It is then noticeable through the previous equation that Kröner–Weng's model is similar to the present model by taking $b^I = 1$. Consequently, the stress and strain concentration tensors of the phase 1 are immediately deduced from (74) and (76) so that

$$\widehat{B}^{\mathrm{KW}^1}(p) = \frac{p + \frac{6}{5}\frac{\mu^e a^2}{2\eta^2}}{\frac{p}{a^1} + \frac{6}{5}\mu^e\left(\frac{1}{2\eta^1} - f_1\left[\frac{a^1}{2\eta^1} - \frac{a^2}{2\eta^2}\right]\right)} \tag{78}$$

and

$$\widehat{A}^{\mathrm{KW}^I}(p) = \frac{\widehat{B}^{\mathrm{KW}^I}(p)\left(p + \frac{\mu^I}{\eta^I}\right)}{2\mu^I\left(a^I\left(\frac{\widehat{B}^{\mathrm{KW}^I}(p)}{2\eta^I} - \frac{\overline{\widehat{B}^{\mathrm{KW}^I}(p)}}{2\eta^I}\right) + \frac{\overline{\widehat{B}^{\mathrm{KW}^I}(p)}}{2\eta^I}\right) + \frac{\mu^I}{\mu^e}p}. \tag{79}$$

4.1.4. Numerical results

In order to compare the three models, the strain concentration tensors of the phase 1 $\widehat{A}^{H^I}$, $\widehat{A}^{\mathrm{KW}^I}$, $\widehat{A}^{\mathrm{PM}^I}$ are plotted as a function of p for three different volume fractions: $f_1 = 0.25$, 0.5 and 0.75. For computations, the elastic and viscous moduli are taken such as

$\mu^1 = 50$ MPa and $\eta^1 = 10$ MPa s for the phase 1;
$\mu^2 = 250$ MPa and $\eta^2 = 1000$ MPa s for the phase 2.

Results obtained by the present model for the three volume fractions (Fig. 1A–C) closely follow those determined from Hashin–Rougier et al.'s model, considered as a reference for the viscoelastic self-consistent model. In particular, asymptotic situations corresponding to pure elasticity (resp. pure viscoplasticity) reached when the physical time $t \to 0$ or the Laplace–Carson parameter $p \to \infty$ (resp. $t \to \infty$ or $p \to 0$) are interesting to analyze. The pure elastic strains are obtained from (67) and (76) when $p \to \infty$. Both models provide the same result that is for example for the phase 1

$$\hat{\varepsilon}^1 = \frac{5\mu^e}{3\mu^e + 2\mu^1}\,\widehat{E}.$$

Similarly, the pure viscoplastic strains are obtained from (67) and (76) when $p \to 0$. Hashin–Rougier's model and the present model tend to the same result that is for the phase 1

$$\hat{\varepsilon}^1 = \frac{5\eta^e}{3\eta^e + 2\eta^1}\,\widehat{E}.$$

In addition to the fact that asymptotic results are identical, the results obtained by both models remain very close for the transient regime even for high mechanical contrast between the two phases. These results that concern the local response strengthen the relevance of the present model, which succeeds in accounting for viscoelastic interactions.

As expected, more discrepancies appear as far as Kröner–Weng's model is concerned; especially when the material tends to a pure viscoplastic behavior ($p \to 0$ in (79)), the strain rate concentration tensor tends to 1. This result is coherent with the uniform viscoplastic strain field hypothesis but does not represent the physical reality.
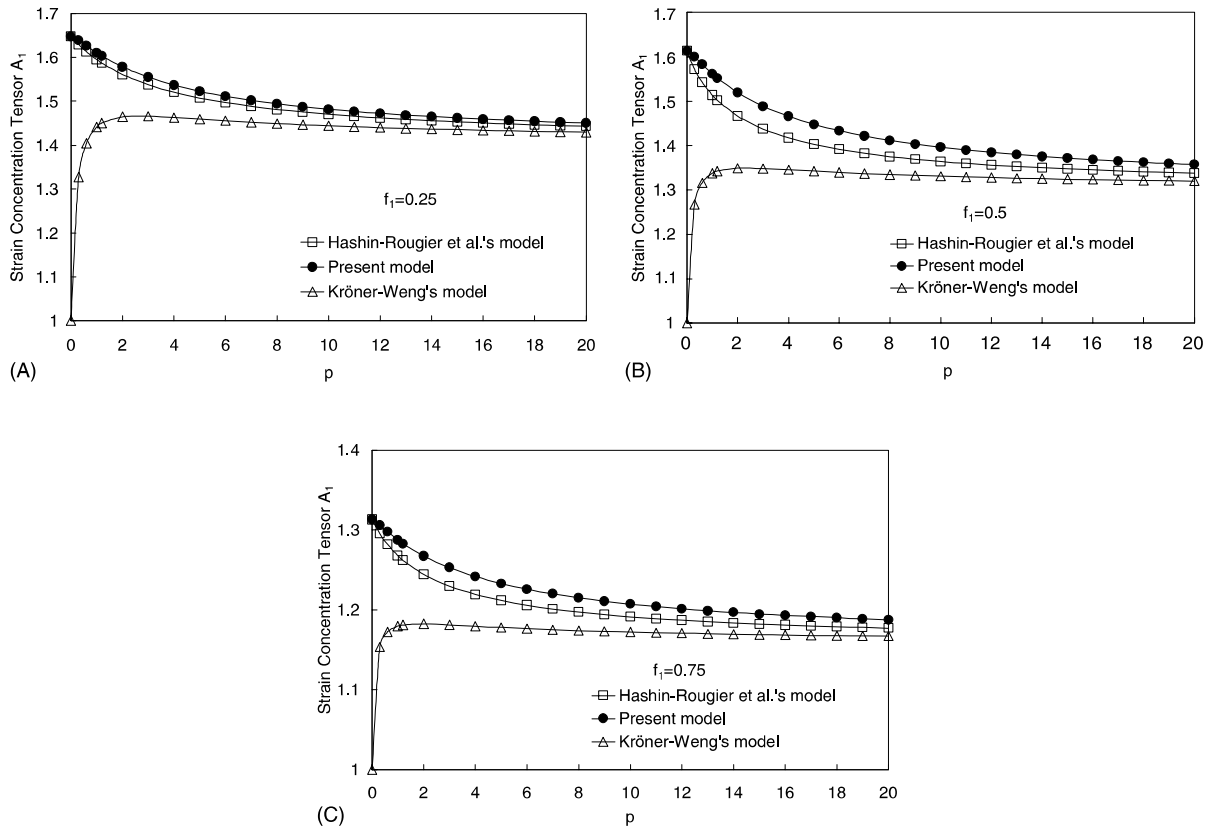
Fig. 1. (A) Strain concentration tensors as a function of p for the phase 1 of a two-phase incompressible and isotropic viscoelastic composite with $f_1 = 0.25$. (B) Strain concentration tensors as a function of p for the phase 1 of a two-phase incompressible and isotropic viscoelastic composite with $f_1 = 0.5$. (C) Strain concentration tensors as a function of p for the phase 1 of a two-phase incompressible and isotropic viscoelastic composite with $f_1 = 0.75$.

4.1.5. Tensile tests and numerical results

Tensile tests have been simulated at a strain-rate of $|\dot{E}| = 10^{-4}$ s^{-1} for the two-phase material previously defined and having a volume fraction of the phase 1 equal to 0.25. The homogenized behavior obviously reflects the preceding results (Fig. 2). A very good agreement between results obtained by Hashin–Rougier's model and the present one is found. In that case, the asymptotic situations are captured at very low strains for the pure elasticity and at large strains for the pure viscoplasticity. As expected from the preceding analyze of the interaction laws, Kröner–Weng's model strongly overestimates stresses in the material, especially at large deformations or, that is equivalent, at long time. It should be underlined that the previous model based on translated fields developed by Paquin et al. (1999) also provides results very close to Hashin–Rougier's model and the present one (Fig. 2). This result shows that the field translation method with respect to compatible ones is all the more relevant than the reference fields have physical sense. It allows then to well capture the intergranular accommodation.

4.2. The case of a polycrystalline elastic–viscoplastic material

The second attempt is to model the overall behavior of a BCC elastic–viscoplastic polycrystal through the present self-consistent model. In the following, the non-linear behavior of the single crystal is described
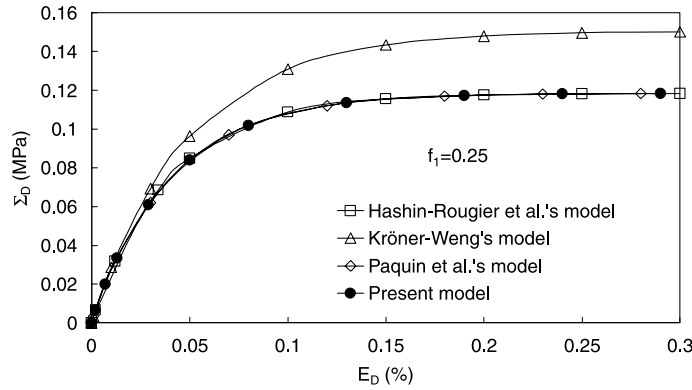
Fig. 2. Deviatoric stress–strain tensile curves at a strain rate of $|\dot{E}| = 10^{-4}$ s^{-1} for a two-phase incompressible and isotropic viscoelastic composite with $f_1 = 0.25$.

in the framework of crystalline plasticity through the classical secant formulation. The overall response obtained with the present model is compared respectively with the models of Kröner–Weng (1961, 1981) and Paquin et al. (1999).

4.2.1. Single crystal behavior

In the following, the inelastic strain is assumed to be only issued from crystallographic slip according to two slip systems $(1\,1\,0)\langle 1\,1\,1 \rangle$ and $(1\,1\,2)\langle 1\,1\,1 \rangle$ related to the BCC structure. The orientation tensor (or Schmid tensor) is then defined for each slip system (s)

$$R_{ij}^{(s)} = \tfrac{1}{2}\left(m_i^{(s)} n_j^{(s)} + m_j^{(s)} n_i^{(s)} \right),$$

where $n^{(s)}$ and $m^{(s)}$ are respectively the unit vector normal to the slip plane and along the slip direction of the slip system (s). The resolved shear stress on each slip system (s) is

$$\tau^{(s)} = R_{ij}^{(s)} \sigma_{ij}$$

and finally the expression of the viscoplastic strain rate tensor is given by

$$\dot{\varepsilon}_{ij}^{\mathrm{vp}} = \sum_s R_{ij}^{(s)} \dot{\gamma}^{(s)}. \tag{80}$$

A local physically based flow rule allows to take into account the strain rate sensitivity of the material so that (Kocks et al., 1975)

$$\dot{\gamma}^{(s)} = \dot{\gamma}_0 \left(\frac{\tau^{(s)}}{\mu} \right)^2 \exp\left(-\frac{\Delta G}{kT} \left(1 - \left(\frac{|\tau^{(s)}|}{\tau_{\mathrm{r}}^{(s)}} \right)^p \right)^q \right) sign(\tau^{(s)}). \tag{81}$$

In (81), $\dot{\gamma}_0$ is a reference strain rate, μ is the elastic shear modulus, k is the Boltzmann constant, T is the absolute temperature and ΔG is the activation energy linked to the activation process. The constant parameters p and q characterize the space distribution obstacles to dislocation motion so that $0 < p \leqslant 1$ and $1 \leqslant q \leqslant 2$. $\tau^{(s)}$ is the resolved shear stress previously defined and $\tau_{\mathrm{r}}^{(s)}$ is a reference shear stress characteristic of strain hardening.

The local viscoplastic strain rate tensor is written according to the secant formulation

$$\dot{\varepsilon}_{ij}^{\mathrm{vp}} = m_{ijkl}^{\mathrm{vp}}(\sigma) \sigma_{kl},$$

where according to (80) and (81)

$$m_{ijkl}^{\text{vp}} = \sum_s \dot{\gamma}_0 \frac{|\tau^{(s)}|}{\mu^2} \exp\left(-\frac{\Delta G}{kT}\left(1 - \left(\frac{|\tau^{(s)}|}{\tau_{\text{r}}^{(s)}} \right)^p \right)^q \right) R_{ij}^{(s)} R_{kl}^{(s)}. \tag{82}$$

The evolution of the reference shear stress present in (81) and (82) is linked to the hardening matrix H^{sh}

$$\dot{\tau}_{\text{r}}^{(s)} = \sum_s H^{\text{sh}} \left| \dot{\gamma}^{(h)} \right|. $$

The strain-hardening model used takes into account both the creation and the annihilation of dislocations so that the hardening matrix becomes

$$H^{\text{sh}} = \frac{\alpha\mu}{2\sqrt{\sum_k a^{sk}\rho^{(k)}}} a^{\text{sh}}\left(\frac{1}{L^{(h)}} - y_{\text{c}}\rho^{(h)} \right). \tag{83}$$

In (83), α is a constant (typically 0.5), $L^{(h)}$ deals with a mean free path of dislocations which decreases as a function of deformation according to the following rule:

$$L^{(h)} = \frac{K}{\sqrt{\sum_{u \neq h} \rho^{(u)}}}$$

y_{c} is a characteristic length describing the annihilation dislocation process, $\rho^{(h)}$ is the dislocation density on the slip system (h) and a^{gh} is the crystallographic part of the hardening matrix as defined by Franciosi (1985). This formulation is advantageous since dislocation densities are used as internal variables and their evolution on the slip system (s) is given by

$$\dot{\rho}^{(s)} = \frac{1}{b}\left(\frac{1}{L^{(s)}} - y_{\text{c}}\rho^{(s)} \right)\left| \dot{\gamma}^{(s)} \right|,$$

where b is the magnitude of the Burgers vector.

4.2.2. Material parameters

The polycrystal is considered as an isotropic aggregate constituted of 100 spherical grains. The elastic behavior is assumed isotropic (shear modulus ($\mu = 80\,000$ MPa) and Poisson ratio ($v = 0.3$)). Because of the incompressibility of the viscoplastic behavior defined in (82), it is necessary like Hutchinson (1976) to add a negligible compressible contribution. This procedure is detailed in Paquin et al. (2001). The material constants are gathered in Table 1.

4.2.3. Tensile stress–strain curves

In this section, Kröner–Weng's model (1961, 1981), Paquin et al.'s model (1999) and the present model are compared in the case of a tensile test with a prescribed strain rate $|\dot{E}| = 10^{-2}$ s^{-1}. The numerical results are represented on the Fig. 3. As expected, the three models provide similar results at very low strains corresponding to the pure elastic response of the material. At large strains, Paquin et al.'s model and the present one tend to the same stress limit whereas Kröner–Weng's model predicts a higher stress level. These

Table 1
Material parameters used for computations

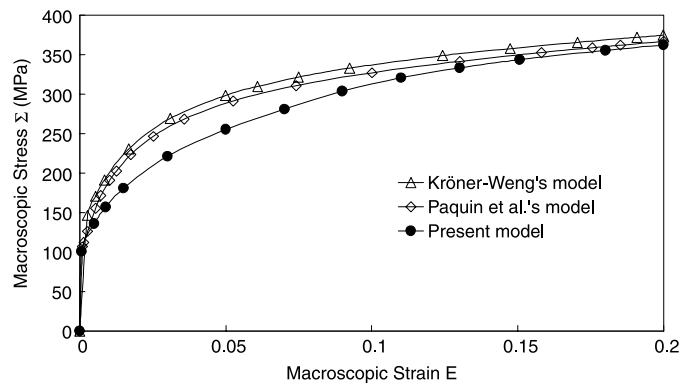| $\dot{\gamma}_0$ (s^{-1})	$\frac{\Delta G}{k}$ (K)	p	q	y_{c} (m)	b (m)	K	α
1.5×10^{-2}	$10\,000$	1	2	8×10^{-8}	2.5×10^{-10}	10	0.5

Fig. 3. Stress–strain tensile curves at a strain-rate of $|\dot{E}| = 10^{-2}$ s^{-1} for a polycrystalline BCC metal of 100 crystals.

expected discrepancies are much lower than those observed in the two-phase material case. This is mainly due to the weaker mechanical contrast between the heterogeneities involved in the polycrystalline BCC metals than in the two-phase material. Indeed, in the case of the BCC polycrystalline metals, the elasticity is usually considered as homogeneous and plastic anisotropy remains low due to the numerous possible slip systems. Fig. 3 exhibits differences between Paquin et al.'s model and the present one in the transient regime. The present model provides a smoother transition from the pure elastic to the pure viscoplastic response and withdraws from results of Kröner–Weng's models. This comes from a different estimation of the elastic–viscoplastic interactions linked to the choice of other translated fields and Navier operators.

5. Conclusion

The complex and strong space–time connections arising from elastic–viscoplastic interactions in heterogeneous materials have been captured through a new class of models involving simultaneously self-consistency for the viscous part and self-equilibrium for the remaining fields. Owing to different orders of time derivation in the local constitutive equation, an internal variable approach is preferred to a hereditary one so that the evolution equations are naturally determined and do not required the knowledge of the whole mechanical history. The use of the projection operator properties eases the introduction of translated fields and, in this case, the pure viscoplastic classical self-consistent solution has been naturally chosen. With this straightforward choice, results obtained for a two-phase composite as well as for a non-linear polycrystal and compared with other models demonstrate the quality and the efficiency of the present formulation. The proposed scheme may also be extended to other materials with more complex physical and mechanical couplings.

References

Berveiller, M., Zaoui, A., 1979. An extension of the self-consistent scheme to plastically-flowing polycrystals. J. Mech. Phys. Solids 26, 325–344.
Berveiller, M., Zaoui, A., 1981. A simplified self-consistent scheme for the plasticity of two-phase metals. Res. Mech. Lett. 1, 119–124.
Berveiller, M., Fassi-Fehri, O., Hihi, A., 1987. The problem of two plastic and heterogeneous inclusions in an anisotropic medium. Int. J. Engng. Sci. 25 (6), 691.
Dederichs, P.H., Zeller, R., 1973. Variational treatment of the elastic constants of disordered materials. Z. Phys. 259, 103–116.

Eshelby, J.D., 1957. The determination of the elastic field of an ellipsoidal inclusion and related problems. Proc. R. Soc. Lond. A 241, 376–396.

Franciosi, P., 1985. The concepts of latent hardening and strain hardening in metallic single crystals. Acta Metall. 33 (9), 1601–1612.

Hashin, Z., 1969. The inelastic inclusion problem. Int. J. Engng. Sci. 7, 11–36.

Hershey, A.V., 1954. The elasticity of an isotropic aggregate of anisotropic cubic crystals. Trans. ASME J. Appl. Mech. 21, 236–240.

Hill, R., 1965. Continuum micromechanics of elastoplastic polycrystals. J. Mech. Phys. Solids 13, 89–101.

Hutchinson, J.W., 1976. Bounds and self-consistent estimates for creep of polycrystalline materials. Proc. R. Soc. Lond. A 348, 101–127.

Kocks, U.F., Argon, A.S., Ashby, M.F., 1975. Thermodynamics and kinetics of slip. In: Progress in Materials Science, vol. 19. Pergamon Press, Oxford.

Kouddane, R., Molinari, A., Canova, G.R., 1993. Self-consistent modelling of heterogeneous viscoelastic and elastoplastic materials. In: Teodosiu, C., Raphanel, J.L., Sidoroff, F. (Eds.), MECAMAT'91, Large Plastic Deformations, Fundamentals and Application to Melal forming, Balkema, Rotterdam, pp. 185–192.

Kröner, E., 1958. Berechnung der elastischen Konstanten des Vielkristalls aus den Konstanten des Einskristalls. Z. Phys., 151.

Kröner, E., 1961. Zur plastischen Verformung des Vielkristalls. Acta Metall. 9, 155–161.

Kröner, E., 1967. Elastic moduli of perfectly disordered composite materials. J. Mech. Phys. Solids 15, 319–329.

Kröner, E., 1977. Bounds for effective elastic moduli of disordered materials. J. Mech. Phys. Solids 25, 137–155.

Kröner, E., 1986. Statistical modelling. In: Gittus, E., Zarka, E. (Eds.), Modelling Small Deformations of Polycrystals. Elsevier Appl. Sci. Pub, London, New York.

Kröner, E., 1989. Modified green functions in the theory of heterogeneous and/or anisotropic linearly elastic media. In: Weng, G.J., Taya, M., Abe, M. (Eds.), Micromechanics and Inhomogeneity. Springer, Berlin, pp. 197–211.

Kunin, I.A., 1983. Elastic Media with Microstructure II: Three Dimensional Models. Springer Series in Solid-State Sciences, vol. 44. Springer Verlag, Berlin, Heidelberg, New York, Tokyo.

Li, J., Weng, G.J., 1994. Strain-rate sensitivity, relaxation behavior, and complex moduli of class of isotropic viscoelastic composites. J. Engng. Mat. Tech. 116, 495–504.

Li, J., Weng, G.J., 1997. A secant-viscosity approach to the time-dependent creep of an elastic–viscoplastic composite. J. Mech. Phys. Solids 45, 1069–1083.

Lin, T.H., 1957. Analysis of elastic and plastic strains of a FCC crystal. J. Mech. Phys. Solids 5, 143–149.

Masson, R., Zaoui, A., 1999. Self-consistent estimates for the rate-dependent elastoplastic behaviour of polycrystalline materials. J. Mech. Phys. Solids 47, 1543–1568.

Mori, T., Tanaka, K., 1973. Average stress in matrix and average elastic energy of materials with misfitting inclusions. Acta Metall. 21, 571–574.

Nemat-Nasser, S., Obata, M., 1986. Rate dependent finite elastoplastic deformation of polycrystals. Proc. R. Soc. Lond. A 407, 343–375.

Paquin, A., Sabar, H., Berveiller, M., 1999. Integral formulation and self-consistent modelling of elastoviscoplastic behavior of heterogeneous materials. Arch. Appl. Mech. 69, 14–35.

Paquin, A., Berbenni, S., Favier, V., Lemoine, X., Berveiller, M., 2001. Micromechanical modeling of the elastic–viscoplastic behavior of polycrystalline steels. Int. J. Plasticity 17, 1267–1303.

Rougier, Y., Stolz, C., Zaoui, A., 1993. Représentation spectrale en viscoélasticité linéaire des matériaux hétérogènes. C. R. Acad. Sci. Paris 316, 1517–1522.

Rougier, Y., Stolz, C., Zaoui, A., 1994. Self-consistent modeling of elastic–viscoplastic polycrystals. C. R. Acad. Sci. Paris 318, 145–151.

Taylor, G.I., 1938. Plastic strain in metals. J. Inst. Metals 61, 307–324.

Toth, L.S., Molinari, A., 1994. Tuning a self-consistent viscoplastic model by finite element results-I, modeling. Acta Metall. Mater. 42 (7), 2453–2458.

Weng, G.J., 1981. A self-consistent scheme for the relaxation behavior of metals. J. Appl. Mech. 48, 779–784.